где C не зависит от δ, а $\lambda = 4/\pi^2$ при $G \in J(q)$, $\lambda = 4$ при $G \in J_0(q)$.

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UPPER ESTIMATES OF AIRFOIL AERODYNAMIC CHARACTERISTICS FOR A VISCOUS INCOMPRESSIBLE FLOW

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In the present work the method of the inverse boundary-value problems is used to obtain upper estimates for two main integral airfoil characteristics: the lift coefficient C_l and the aerodynamic ratio K in a certain class of airfoils flown by an incompressible flow of a viscous fluid at a fixed angle of attack at high Reynolds number $(10^6 \div 10^7)$ with the presence of a thin non-separating turbulent boundary layer.

1. Introduction. Problems of airfoil shape optimization to attain extremal aerodynamic characteristics are the subject of special interest for researchers working in the field of applied and theoretical hydro- and gasdynamics [1, 2]. One of the approaches, allowing to give correct variational formulations and obtain explicit solutions of airfoil optimizations problems, is connected with the theory of inverse boundary-value problems [3, 4]. Its efficiency is determined by the fact that in the reference of certain models of flow around an airfoil its main integral characteristics, such as the lift C_l, the drag C_d, can be expressed through some control function, connected with the shape of optimized airfoil. From practical point of view high values of the lift coefficient C_l allow to provide good manoeuvring performances and low landing speed. High values of the aerodynamic ratio $K = C_l/C_d$ (C_d is the drag coefficient) provides the long flight range. Modern optimized theoretical airfoils provide the value of the ratio about $80 \div 120$, at the condition that the boundary layer on the airfoil is non-separating and fully turbulent (see [5]).

In the present paper the method of the inverse boundary-value problems is used to obtain upper estimates of the lift coefficient and the aerodynamic ratio as the explicit solutions of two variational problems of airfoil theory: maximisation of the lift C_l, and maximisation of the ratio K in a certain class of airfoils flown by an incompressible flow of viscous fluid for a fixed angle of attack at high Reynolds number ($10^6 \div 10^7$) with the presence of a thin non-separating turbulent boundary layer.

2. Basic equations. In $z = x + iy$-plane we consider a steady non-separating incompressible flow about an impermeable airfoil with smooth boundary and cusped trailing edge B (z=0) with external angle $\pi\varepsilon$, $2 \geq \varepsilon \geq 1$ (Fig. 1, a), the x-axis is assumed to be parallel with the freestream direction.

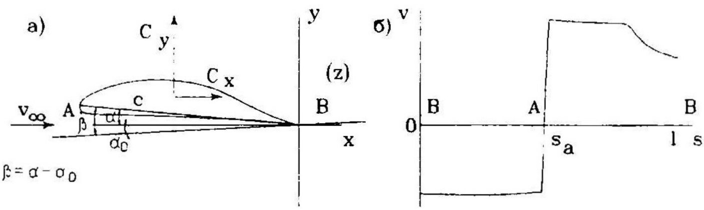

Fig. 1. Problem formulation

We also assume the airfoil contour perimeter l to be of unit length ($l = 1$). The flow at infinity is uniform with freestream velocity $v_\infty = 1$. The Reynolds number $Re = v_\infty l/(2\nu)$ (ν is the kinematic viscosity) is such high that in the range of non-separating flow the viscosity is manifesting only in a thin turbulent boundary layer near the airfoil. The boundary layer thickness is assumed negligible small.

Usually the dimensionless parameters (Re, lift coefficient C_l, drag coefficient C_d etc.) are related to the airfoil chord c. In the present problem it is convenient to relate them to the half of perimeter, which for real airfoil is close to the chord length $l/2 \approx c$. We introduce basic relationship following [3].

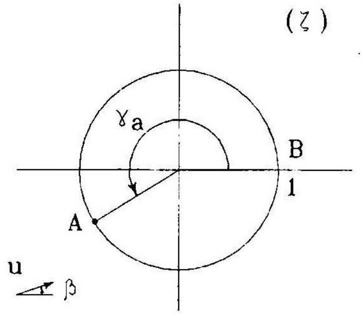

Fig. 2. Auxiliary plane

We consider auxiliary domain $E = \{\zeta :\mid \zeta \mid \geq 1\}$ (Fig. 2) in $\zeta = re^{i\gamma}$-plane, and introduce a class of incompressible flows indicated above through a real, 2π-periodic, Hölder continuous control function $S(\gamma) \in H^\alpha[0, 2\pi]$, $\alpha \in (0, 1]$, such that

$$J_0(S) = \int_0^{2\pi} S(\tau)d\tau = 0 \qquad (1)$$

where γ is polar angle in ζ-plane. The integral equation guarantees the unit freestream velocity.

The function $z(\zeta)$ realising conformal mapping of E onto the flow region so that $z(\infty) = \infty$, $z(1) = 0$ is of the form

$$z(\zeta) = u\int_1^\zeta e^{-\chi(\zeta)}(1 - \zeta^{-1})^{\varepsilon - 1}d\zeta, \qquad \chi(\zeta) = -\frac{1}{2\pi}\int_0^{2\pi} S(\tau)\frac{e^{i\tau} + \zeta}{e^{i\tau} - \zeta}d\tau, \quad (2)$$

where u is a constant, determining linear scale in the z-plane. The airfoil contour co-ordinates x, y and the velocity distribution v, when the airfoil chord is declined for angle β from the zero-lift direction are determined by the following parametrical formulae

$$x(\gamma) = \int_0^\gamma \frac{ds}{d\gamma}(\tau)\cos\theta(\tau)d\tau, \qquad y(\gamma) = \int_0^\gamma \frac{ds}{d\gamma}(\tau)\sin\theta(\tau)d\tau,$$

$$v(\gamma,\beta) = \exp[S(\gamma)]|2\cos(\gamma/2-\beta)||2\sin(\gamma/2)|^{2-\varepsilon}. \qquad (3)$$

Here s is the airfoil contour arc length and $\theta(\gamma)$ is the tangent angle of the contour

$$\frac{ds}{d\gamma} = -ue^{-S(\gamma)}|2\sin(\gamma/2)|^{\varepsilon-1},$$

$$\theta(\gamma) = -\frac{1}{2\pi}\int_0^{2\pi} S(\tau)ctg\frac{\tau-\gamma}{2}d\tau + \frac{(3-\varepsilon)\gamma}{2} + \frac{\varepsilon\pi}{2}.$$

The angle of attack α is determined as the angle between airfoil chord and x-axis. The value of u is calculated from the condition that the perimeter is unit

$$u = [I_1(S)]^{-1}, \qquad I_1(S) = \int_0^{2\pi} |2\sin(\tau/2)|^{\varepsilon-1}e^{-S(\tau)}d\tau.$$

Airfoil contour closedness is guaranteed, if the integral equalities are fulfilled

$$J_1(S) = \int_0^{2\pi} S(\tau)\cos\tau d\tau + (\varepsilon-1)\pi = 0, \ J_2(S) = \int_0^{2\pi} S(\tau)\sin\tau d\tau = 0. \quad (4)$$

The lift coefficient is determined by the Zhukovskiy formula

$$C_l = 16\pi\sin\beta/I_1(S). \qquad (5)$$

The value of β is usually called the theoretical angle of attack and characterises the deviation of the airfoil chord from the zero lift direction. In particular, for $\beta = 0$ from (5) we have $C_l = 0$. Moreover, from (3) it follows that the leading critical point A corresponds to a point on the circle with polar angle $\gamma = \gamma_a = \pi + 2\beta$.

The drag coefficient, as a rule, is obtained on the base of an integral method for boundary-layer calculation in the form of a complicated integro-differential functional $C_d = \mathbf{F}(S, Re, \beta)$. In particular, in the case of fully attached turbulent boundary layer the Squire-Young formula gives one of simple variants for drag coefficient calculation (cf. [6, p. 623])

$$C_d = 2^\mu A(Re)^{-\nu}\left[\left(\int_{s_a}^{1} v^\eta(s)ds\right)^\mu + \left(\int_0^{s_a} v^\eta(s)ds\right)^\mu\right]$$

where $v(s)$ is the airfoil velocity distribution as function of the arc length s (Fig. 1, b); A, ν, μ, η are interconnected empirical constants. In particular, in [6] the following two choices are described

$$A = 0,0307, \quad \mu = 6/7, \quad \nu = 1/7, \quad \eta = 4, \qquad Re \geq 5 \cdot 10^6, \tag{6}$$

$$A = 0,0704, \quad \mu = 4/5, \quad \nu = 1/5, \quad \eta = 7/2, \qquad Re < 5 \cdot 10^6. \tag{7}$$

Another slightly different choices of the constants was used in [7]. After some algebra we rewrite the expression through control function in the form

$$C_d(\beta) = A(2u)^{\mu}(Re)^{-\nu}\left[R_1^{\mu}(\beta) + R_2^{\mu}(\beta)\right]$$

where

$$R_1(\beta) = \int\limits_0^{\pi+2\beta} e^{(\eta-1)S(\tau)}\,|2\cos(\tau/2-\beta)|^{\eta}\,|2\sin(\tau/2)|^{\lambda}d\tau,$$

$$R_2(\beta) = \int\limits_{\pi+2\beta}^{2\pi} e^{(\eta-1)S(\tau)}\,|2\cos(\tau/2-\beta)|^{\eta}\,|2\sin(\tau/2)|^{\lambda}d\tau,\ \lambda = (2-\varepsilon)\eta + \varepsilon - 1.$$

Then for the aerodynamic ratio we have

$$K = C_l/C_d = \frac{8\pi(Re)^{\nu}(2I_1)^{\mu-1}\sin\beta}{A[R_1^{\mu}(\beta) + R_2^{\mu}(\beta)]}.$$

We simplify the functional noting that $\mu \approx 1$. We assume $\mu = 1$. Then expression for the aerodynamic ratio takes the form

$$K = \frac{8\pi(Re)^{\nu}\sin\beta}{AI_2(S,\beta)}, \tag{8}$$

$$I_2(S,\beta) = \int\limits_0^{2\pi} e^{(\eta-1)S(\tau)}|2\cos(\tau/2-\beta)|^{\eta}|2\sin(\tau/2)|^{\lambda}d\tau.$$

In Fig. 3 we presented some comparison of the results obtained from the simplified formula and results, obtained from the boundary layer calculation method from [4] for NACA 23015. The angle of attack was supposed to be varied in the interval $-5^o \leq \alpha \leq 5^o$. The corresponding velocities distributions are presented in the Fig. 3, a. In Fig. 3, b we present the dependencies $C_l(C_d)$ obtained by Eppler method for $Re = 5 \cdot 10^5$ (solid line),

$Re = 10^6$ (dashed line), $Re = 5 \cdot 10^6$ (dashed and dotted line). Here we also present the corresponding relationships $C_l(C_d)$ calculated from (5) and (8) for $Re = 5 \cdot 10^5$ (rombs), $Re = 10^6$ (rectangles), $Re = 5 \cdot 10^6$ (triangles). The maximal error is not more that 5%. This and other comparisons has shown that for slightly cambered airfoils the formulae (5), (8) let obtain a rather reliable results.

Thus, for a fixed β the problem of maximisation of C_l and K is reduced to minimization of the functionals $I_1(S)$ and $I_2(S;\beta)$ in the convex set of Hölder continuous functions (see (1), (4))

$$U = \{S(\gamma) \in H^\alpha[0, 2\pi] : J_0(S) = 0, J_1(S) = \pi(1-\varepsilon), J_2(S) = 0\},$$

which is a compact in $L_2[0, 2\pi]$.

3. Explicit solutions of the variational problems. The following theorem is valid.

Theorem 1. *The functionals $I_1(S)$ and $I_2(S;\beta)$ (for fixed β) are strictly convex on the convex compact $U \subset L_2[0, 2\pi]$ and, consequently, problems of minimisation of $J(S)$ and $I_2(S;\beta)$ in U are uniquely solvable.*

Proof. Since $I_1(S)$ is twice differentiable, the strict convexity of $I_1(S)$ is provided iff the following the known criterion is fulfilled

$$\langle (I_1'' P)\xi, \xi \rangle > 0 \quad \text{for all} \quad 0 \not\equiv \xi(\gamma) \in L_2[0, 2\pi], \quad P(\gamma) \in U.$$

Here $\langle (I_1'' P)\xi, \xi \rangle$ denotes the value of the linear functional $(I_1'' P)(\xi)$ for $\xi \in L_2[0, 2\pi]$. We have

$$\langle (I_1'' P)\xi, \xi \rangle = \int_0^{2\pi} \exp[P(\tau)][2\sin(\tau/2)]^{\varepsilon-1}\xi^2(\tau)d\tau > 0.$$

The set U is convex and bounded in the space of Hölder functions $H^\alpha[0, 2\pi]$. Consequently, U is convex and compact in $L_2[0, 2\pi]$. Now the statement of theorem can be obtained as consequence of Weierstrass theorem.

Proof of strict convexity of functional $I_2(S;\beta)$ (for fixed β) is carried out analogously.

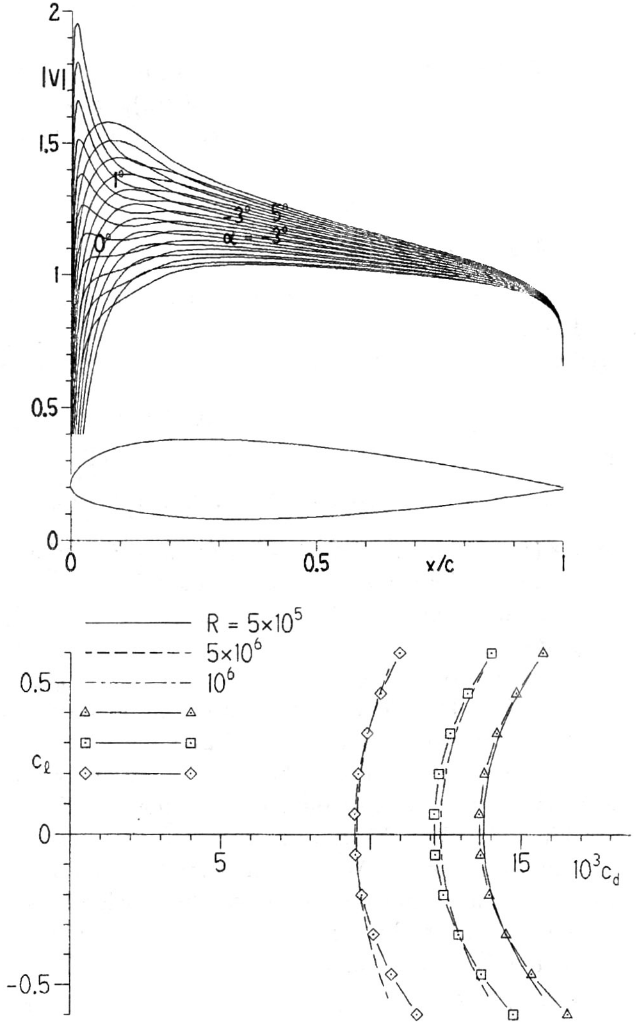

Fig. 3. NACA 23015

We consider problems of minimisation of functionals $I_1(S)$ and $I_2(S;\beta)$ in the linear subspace $U_0 \subset L_2[0,2\pi]$, determined by (1) и (4). Then the following theorems are valid.

Theorem 2. *Functional $I_1(S)$ attains the global minimum in $U_0 \subset L_2[0,2\pi]$ at the point $S_*(\gamma) = (\varepsilon - 1)\ln[2\sin(\gamma/2)]$, where $I_1(S_*) = 2\pi$. Therefore in the framework of the accepted model of flow the maximum of the lift is $C_{l*} = 8\sin\beta$.*

Proof. We compose an extended functional

$$\Psi(S) = I_1(S) + \mu_0 J_0(S) + \mu_1 J_1(S) + \mu_2 J_2(S)$$

where μ_0, μ_1, μ_2 are unknown Lagrange multipliers. Necessary condition for extremum of $\Psi(S)$ in $L_2[0,2\pi]$ is equivalent to the following relationship, which must be valid for an arbitrary element of $\xi \in L_2[0,2\pi]$

$$\int_0^{2\pi} \left\{-\exp[-S(\tau)]|2\sin(\tau/2)|^{\varepsilon-1} + \mu_0 + \mu_1\cos\tau + \mu_2\sin\tau\right\}\xi(\tau)d\tau = 0.$$

This implies

$$S_*(\gamma) = -\ln\left(\mu_0 + \mu_1\cos\gamma + \mu_2\sin\gamma\right) + (\varepsilon - 1)\ln\left(2\sin(\gamma/2)\right).$$

The conditions (1) and (4) implies $\mu_0 = 1$, $\mu_1 = \mu_2 = 0$. Strict convexity of $I_1(S)$ guarantees the uniqueness of the solution obtained. Direct integration gives $I_1(S_*) = 2\pi$ and, consequently, the maximum of lift the $C_{l*} = 8\sin\beta$. Theorem is proved.

For $\varepsilon = 1$ function $S_* \equiv 0 \in U$, and maximum of lift is attained for a disk of $1/(2\pi)$ radius. The global maximum of the lift corresponds to the case $\beta = \pi/2$ (trailing and leading critical points are coincident) and is equal to $C_l^* = 8$. It should be noted that $\varepsilon = 1$ is the only case when the optimal solution lies in the chosen class of functions U. For other angles $\pi\varepsilon$ of the trailing edge $S_* \notin U$ and, consequently, there exists no optimal airfoil with smooth boundary. At the same time the maximal value of C_{l*} obtained above does not depend on ε and can be considered as a universal sharp upper estimate of the lift for the chosen class of airfoil.

Similar theorem is valid for functional $I_2(S,\beta)$.

Theorem 3. *The functional $I_2(S,\beta)$ attains global minimum in $U_0 \subset L_2[0,2\pi]$ at the point*

$$S_*(\gamma) = \frac{2}{\eta - 1}\ln|\mu_0 + \mu_1\cos\gamma + \mu_2\sin\gamma| -$$

$$\frac{\lambda}{\eta-1}\ln[2\sin(\gamma/2)] - \frac{\eta}{\eta-1}\ln|2\cos(\gamma/2-\beta)|. \tag{9}$$

where

$$\mu_0 = 1+\eta^2\sin^2\beta, \qquad \mu_1 = -\eta\sin^2\beta, \qquad \mu_2 = \eta\sin 2\beta.$$

Corresponding value of $I_2(S_*\beta) = 2\pi\left(1+\eta^2\sin\beta^2\right)$. *Therefore in the framework of the accepted model of flow the maximum of aerodynamic ratio is*

$$K_*(\beta) = \frac{4Re^{\nu}\sin\beta}{A(1+\eta^2\sin^2\beta)}. \tag{10}$$

Proof. Similarly to the previous theorem we compose extended functional

$$\Psi(S) = I_2(S;\beta) + \mu_0 J_0(S) + \mu_1 J_1(S) + \mu_2 J_2(S)$$

where μ_0, μ_1, μ_2 are unknown Lagrange multipliers. Necessary condition for extremum of functional $\Psi(S)$ in $L_2[0, 2\pi]$:

$$\int_0^{2\pi}[(\eta-1)\exp[(\eta-1)S(\tau)]|2\sin(\tau/2)|^{\lambda}|2\cos(\tau/2-\beta)|^{\eta}-$$

$$-(\mu_0+\mu_1\cos\tau+\mu_2\sin\tau)]\xi(\tau)d\tau = 0.$$

This implies representation (9) for S_*. We fix the unknown coefficients to satisfy (1) and (4). Let us consider the complex analytical function

$$\chi(\zeta) = 2(\eta-1)^{-1}\ln(1-\varrho e^{i\nu}/\zeta) - \lambda(\eta-1)^{-1}\ln(1-\zeta^{-1})-$$

$$-\eta(\eta-1)^{-1}\ln(1+e^{2i\beta}/\zeta) \tag{11}$$

where $\varrho^2 = \mu_0 - 1$, $-2\varrho\cos\nu = \mu_1$, $-2\varrho\sin\nu = \mu_2$. Let $S_*(\gamma) = \Re\chi(e^{i\gamma})$, and $\Re$ denotes the real part of χ (this function is of the form (9)). We see that (1), which is equivalent to $\chi(\infty) = 0$, is fulfilled. To satisfy the closedness conditions we substitute (11) in (2) and obtain

$$z(\zeta) = u\int_1^{\zeta}(1-\varrho e^{i\nu}/\zeta)^{-\frac{2}{(\eta-1)}}(1-\zeta^{-1})^{\frac{\lambda}{\eta-1}+\varepsilon-1}(1+e^{2i\beta}/\zeta)^{\frac{\eta}{\eta-1}}d\zeta. \tag{12}$$

We expand the integrand in the point $\zeta = \infty$ and after integrating we obtain condition of the closedness

$$\varrho e^{i\nu} = \frac{\eta}{2}\left(1-e^{2i\beta}\right). \tag{13}$$

Thus, for the closedness of airfoil contour it must be

$$\mu_0 = 1 + \eta^2 \sin^2 \beta, \qquad \mu_1 = -\eta \sin^2 \beta, \qquad \mu_2 = \eta \sin 2\beta.$$

Finally, we obtain the minimum of functional

$$I_2(S_* \beta) = 2\pi \left(1 + \eta^2 \sin^2 \beta\right)$$

Correspondingly for the maximal value of aerodynamic ratio we obtain (10). Theorem is proved. It is interesting to note that the extremum does not depend on ε.

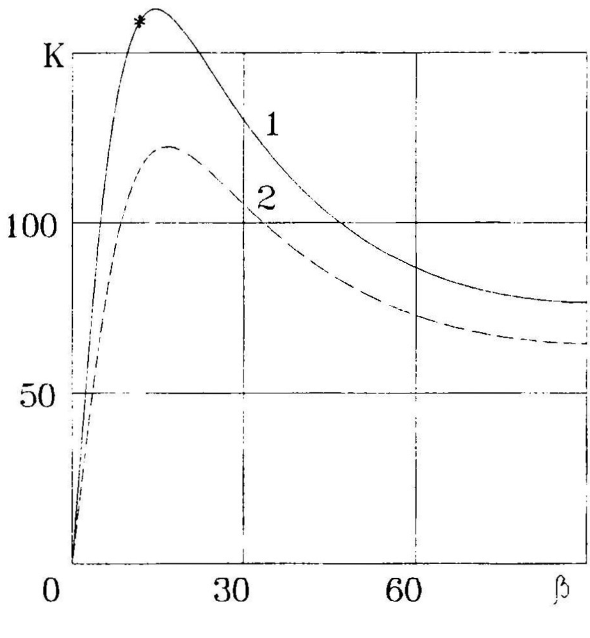

Fig. 4. Maximal aerodynamic ratio vs β

In Fig. 4 dependence (10), calculated for (6) (the solid line 1, $Re = 10^7$) and for (7) (the dashed line 2, $Re = 10^6$) are presented. We see that the functions are not monotone and have absolute maximums $K^* = 163$ at

$\beta_{max} = 11,5^o$ (line 1) and $K^* = 122$ at $\beta_{max} = 12,6^o$ (line 2). This brings us to the following

Sequence. *In the reference of the accepted model of flow the absolute maximum of the aerodynamic ratio is equal to*

$$K^* = 2Re^{\nu}/(\eta A).$$

Proof. It is sufficiently to note that the function $K_*(\beta)$ possesses the absolute maximum which is attained for $\beta = arcsin\sqrt{1/\eta}$.

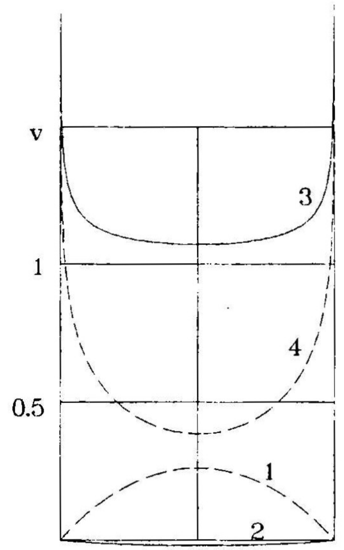

Fig. 5. Optimal "airfoil"

Remark. *It should be noted that for $\beta > \beta_{max}$ the solution of the variational problem possess one additional singularity in the point $\zeta = \rho e^{i\nu}$ (see (13)), which is inside the flow region. Thus solution of the variational* *problem for these values of β does not provide the flow without singularities and should be excluded from the consideration.*

We see that the extremal solution for the aerodynamic ratio does not depend on the angle $\pi\varepsilon$ in the trailing edge. Similarly to the case of lift maximization $S_* \notin U$, what means that the "optimal airfoils" have non-smooth boundary. We designed an "airfoil", corresponding to a point on the extremal curve $K_*(\beta)$ (in Fig. 4 this is the asterisk). For given β its co-ordinates can be determined from (12). The contour obtained for $\beta = 10^o$ is presented in Fig. 5. It has vertical symmetry axis and its "lower surface" (dashed line 1) goes above the "upper surface" (solid line 2). Corresponding velocity distribution $v(x)$ is shown in the upper part of Fig. 5. Here the solid line 3 corresponds to the velocity distribution along upper surface, the dashed line 4 corresponds to velocity on the lower surface. The velocity on the contour is infinite in the leading point (A) and in the trailing edge (B). We see that physically unreliable solutions correspond to the points of the extremal curve $K_*(\beta)$. This can be explained by the fact that conditions of airfoil contour non-intersections are not taken into account during the search of optimal solution. Thus, the present solution gives an upper estimate of aerodynamic ratio for real airfoils in a viscous nonseparating flow at high Reynolds number.

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КРЫЛОВЫЕ ПРОФИЛИ ПОСТОЯННОЙ СКОРОСТИ

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При проектировании крыловых профилей одной из важных проблем является максимизация коэффициента подъемной силы $C_y = 2Y/(L\rho_\infty v_\infty^2)$, где Y – подъемная сила, L – периметр контура профиля, ρ_∞ – плотность жидкости на бесконечности, v_∞ – скорость набегающего потока. Известно (см., например, [1] – [3]), что по модели идеальной несжимаемой жидкости максимальное значение $C_y = 2$ достигается при обтекании круга с совпадающими точками разветвления и схода потока. Несмотря на то, что в последние годы широкое развитие получили численные методы решения задач аэродинамической оптимизации, также представляет интерес изучение класса задач, в которых оптимальные профили удается найти аналитически.

1. Профили с постоянной скоростью на каждой из поверхностей. Пусть искомый профиль обтекается установившимся потоком идеальной несжимаемой жидкости с заданной на бесконечности скоростью v_∞, параллельной оси абсцисс. Требуется найти форму крылового профиля и коэффициент подъемной силы, если скорости на верхней и нижней поверхностях постоянны и равны v_1 и v_2 соответственно ($v_1 > v_2$). Длина верхней части контура равна L. Точка A – особая: в ней скорости терпят разрыв, она также является точкой разветвления потока.

В силу симметрии исходных данных картина течения будет симметрична относительно оси ординат, поэтому достаточно рассмотреть ее левую половину G_z (рис. 1).